

A Hamiltonian Phase-Space Approach to Foreign Exchange: Physics-Based Predictive Distributions without Curve Fitting

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Abstract

We introduce a novel Hamiltonian framework for short-horizon foreign exchange (FX) dynamics that treats statistical moments as position coordinates in a low-dimensional phase space. The system evolves via physically motivated energy terms rather than curve fitting: potential energy encodes mean-reversion through Ornstein–Uhlenbeck (OU) springs with optional cross-couplings, while triangular no-arbitrage constraints act as energy conservation laws. After standardizing to empirical- σ units, we employ Riemannian Manifold Hamiltonian Monte Carlo (RMHMC) to generate predictive distributions and make decisions at the distribution level using CDF-difference statistics and cost-exceedance gating. This approach preserves a “physics-first” philosophy, avoids traditional curve fitting, and explicitly accounts for trading frictions. We demonstrate the theoretical foundations and present the mathematical framework, establishing a new paradigm that bridges statistical mechanics and quantitative finance.

Keywords Hamiltonian dynamics; foreign exchange; Ornstein–Uhlenbeck; Riemannian Manifold HMC; Generalized Hyperbolic distributions; no-arbitrage constraints; standardized coordinates; cost-aware trading.

1 Introduction

The application of physical principles to financial markets has a rich history, from Bachelier’s pioneering work on Brownian motion to modern econophysics [3]. However, most approaches either focus on microscopic order-book mechanics or employ coarse thermodynamic metaphors without establishing rigorous connections to observable market statistics.

We present a fundamentally different approach: a Hamiltonian framework that operates directly on the evolution of statistical moments—mean, standard deviation, skewness, and kurtosis—treating these as position coordinates in a low-dimensional phase space. This choice is motivated by several considerations: (1) moments capture the essential distributional characteristics relevant to trading decisions, (2) they provide a natural bridge between raw price data and theoretical distribution families, and (3) their evolution can be modeled using well-established physical principles.

Our key contributions are:

1. A rigorous mathematical framework treating financial statistics as dynamical variables in a Hamiltonian system

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2. A standardization procedure that enables robust comparison between empirical and theoretical distributions
3. A decision-making protocol based on distributional properties rather than point predictions
4. Integration of no-arbitrage constraints as energy conservation laws
5. A cost-aware gating mechanism that explicitly accounts for trading frictions

The approach deliberately avoids curve fitting and traditional signal calibration, instead relying on physically motivated evolution equations and distribution-level decision criteria. This paradigm shift from prediction to mechanistic understanding offers potential advantages in terms of robustness and theoretical grounding.

2 Related Work and Background

2.1 Econophysics and Hamiltonian Market Models

The intersection of physics and finance has produced numerous insights, from scaling laws in market volatility to applications of statistical mechanics in price formation [3]. However, most Hamiltonian analogies in finance operate at either the microscopic level (modeling individual trades or agent interactions) or employ thermodynamic metaphors without establishing clear connections to observable market quantities.

Our approach differs by defining a compact, observable state space on return moments and specifying energy functions directly in this space. This bridges the gap between abstract physical analogies and practical market modeling.

2.2 Heavy-Tailed Return Modeling

The Generalized Hyperbolic (GH) family and related Lévy processes have become standard tools for modeling financial returns due to their ability to capture heavy tails, skewness, and clustering behavior [1, 2, 4]. Traditional approaches fit these distributions to historical data, but parameter stability remains a persistent challenge.

Our standardization to empirical- σ units represents a novel approach to this problem: rather than fitting model parameters to match data scaling, we transform both empirical and theoretical distributions to a common coordinate system. This preserves distributional comparisons while reducing sensitivity to scale variations.

2.3 Geometric Monte Carlo Methods

Riemannian Manifold Hamiltonian Monte Carlo (RMHMC) [6] extends traditional HMC by incorporating the geometric structure of parameter spaces. While RMHMC has been applied in hierarchical modeling and some financial contexts, its use for one-step predictive evolution in standardized coordinates coupled to distribution-level decisions represents a novel application.

2.4 No-Arbitrage Constraints

Triangular arbitrage constraints in FX markets are well-established [5], typically enforced through direct price relationships. Our treatment as soft energy penalties within a Hamiltonian framework provides a natural mechanism for incorporating these constraints while maintaining the flexibility needed for short-term dynamics.

3 Mathematical Framework

3.1 State Space Definition

Let $\{r_t\}$ denote a time series of returns computed over a rolling window of N observations (e.g., $N = 96$ fifteen-minute bars). We define the moment state vector:

$$q(t) = [\mu(t), \sigma(t), \text{skew}(t), \text{kurt}(t)]^\top \quad (1)$$

where each component represents the corresponding sample moment of recent returns. The conjugate momentum vector captures the rate of change of these statistical properties:

$$p(t) = \dot{q}(t) \approx \frac{q(t) - q(t - \Delta)}{\Delta} \quad (2)$$

where Δ represents the sampling interval in appropriate time units.

3.2 Hamiltonian Structure

The total energy of the system consists of kinetic and potential components:

$$H(q, p) = T(p) + V(q) \quad (3)$$

3.2.1 Kinetic Energy

To ensure numerical stability across different scales, we employ a weighted kinetic energy:

$$T(p) = \frac{1}{2} \sum_{d=1}^4 w_d p_d^2 \quad (4)$$

$$w_d = (\text{Var}[p_d] + \varepsilon)^{-1} \quad (5)$$

where $\text{Var}[p_d]$ is computed from recent momentum history and $\varepsilon > 0$ is a regularization parameter.

3.2.2 Potential Energy

The potential energy encodes mean-reversion dynamics and constraint enforcement:

$$V(q) = V_{\text{self}}(q) + V_{\text{bar}}(q) + V_{\text{arb}}(q) \quad (6)$$

Mean Reversion: Drawing inspiration from Ornstein-Uhlenbeck processes [7], we model mean reversion as elastic restoring forces:

$$V_{\text{self}}(q) = \frac{1}{2} (q - q^{\text{eq}})^\top K (q - q^{\text{eq}}) \quad (7)$$

$$K = \text{diag}(k_\mu, k_\sigma, k_{\text{sk}}, k_{\text{ku}}) + \Delta \quad (8)$$

where q^{eq} represents the equilibrium state (computed from long-term statistics), $k_d \approx 1/\tau_d$ with τ_d being characteristic timescales, and Δ represents optional cross-coupling terms.

Domain Barriers: Physical constraints ensure statistical validity:

$$V_{\text{bar}}(q) = -c_\sigma \log(\sigma - \sigma_{\min}) - c_{\text{ku}} \log(\text{kurt} - 1) \quad (9)$$

enforcing $\sigma > \sigma_{\min} > 0$ and $\text{kurt} > 1$.

No-Arbitrage Constraint: For currency triplets (A, B, C) , triangular arbitrage is penalized through:

$$V_{\text{arb}} = \frac{\eta}{2} \sum_{\Delta} (\log S_{AB} + \log S_{BC} + \log S_{CA})^2 \quad (10)$$

where S_{XY} denotes exchange rates and $\eta > 0$ controls constraint strength.

The full potential is $V = V_{\text{self}} + V_{\text{bar}} + V_{\text{arb}}$.

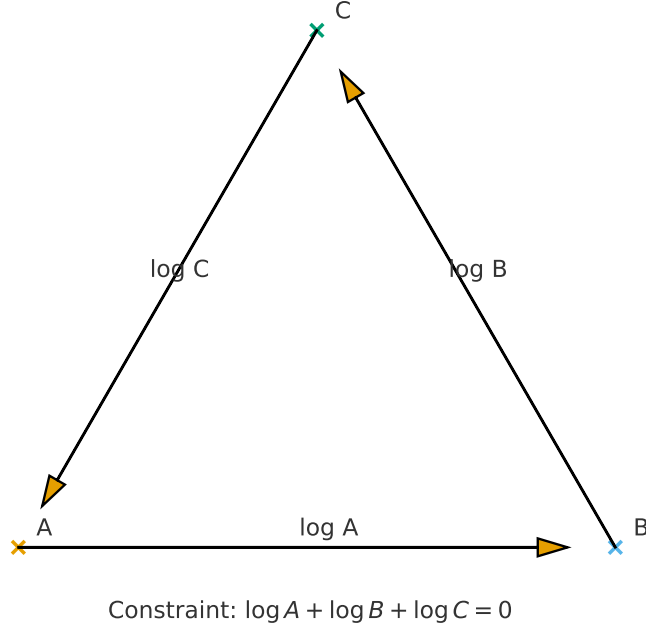


Figure 1: Triangular arbitrage constraint as a soft energy penalty. The framework treats violations of the no-arbitrage condition $\log S_{AB} + \log S_{BC} + \log S_{CA} = 0$ as contributing to the system's potential energy, creating restoring forces that naturally guide the system toward arbitrage-free states while allowing temporary deviations during market transitions.

3.3 Hamilton's Equations

System evolution follows the canonical equations:

$$\frac{dq}{dt} = \frac{\partial H}{\partial p} = W^{-1}p \quad (11)$$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q} = -\nabla_q V(q) \quad (12)$$

where $W = \text{diag}(w_1, w_2, w_3, w_4)$.

4 Standardized Coordinate System

A crucial innovation is the standardization procedure that aligns empirical and theoretical distributions on a common scale. Let σ_{emp} denote the empirical standard deviation of recent returns. We define the standardized coordinate:

$$Z = \frac{X}{\sigma_{\text{emp}}} \quad (13)$$

For Generalized Hyperbolic distributions with parameters $(\mu, \delta, \alpha, \beta, \lambda)$, the transformation to Z -space yields:

$$\mu_Z = \mu / \sigma_{\text{emp}} \quad (14)$$

$$\delta_Z = \delta / \sigma_{\text{emp}} \quad (15)$$

$$\alpha_Z = \alpha \cdot \sigma_{\text{emp}} \quad (16)$$

$$\beta_Z = \beta \cdot \sigma_{\text{emp}} \quad (17)$$

$$\lambda_Z = \lambda \quad (18)$$

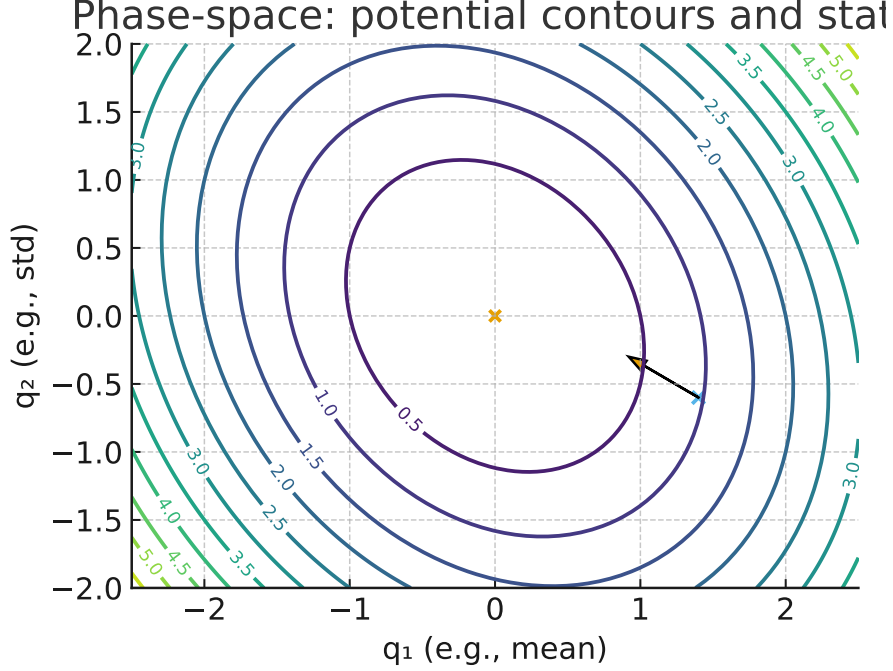


Figure 2: Phase space representation showing potential energy contours in a two-dimensional slice of the moment state space. The contour lines represent constant energy levels, with the equilibrium point (marked by $\star$) at the center of the potential well. The current system state and its momentum vector (arrow) indicate the direction of evolution according to Hamilton’s equations. The spiral trajectory illustrates how the system orbits around equilibrium while conserving total energy.

This standardization ensures that empirical and theoretical distributions operate on comparable scales, facilitating robust distributional comparisons.

5 Predictive Distribution Generation

5.1 Riemannian Manifold HMC

Given current state (q_t, p_t) , we use RMHMC to sample from the predictive distribution at $t + 1$.

The Riemannian metric tensor accounts for the curvature of the parameter space:

$$G_{ij}(q) = -\frac{\partial^2 \log \pi(q)}{\partial q_i \partial q_j} \quad (19)$$

where $\pi(q)$ represents the target distribution.

The RMHMC algorithm generates samples $\{q_{t+1}^{(i)}\}_{i=1}^N$ representing possible future states. Each sample corresponds to a point in the standardized Z -space where distributional comparisons occur.

5.2 Direction Statistic

For decision making, we compute a direction statistic based on CDF differences. Let F_e and F_h denote the empirical and Hamiltonian-evolved CDFs respectively, evaluated at a common pivot point z_0 (typically the standardized median):

$$p_{\text{dir}} = 0.5 + (F_e(z_0) - F_h(z_0)) \quad (20)$$

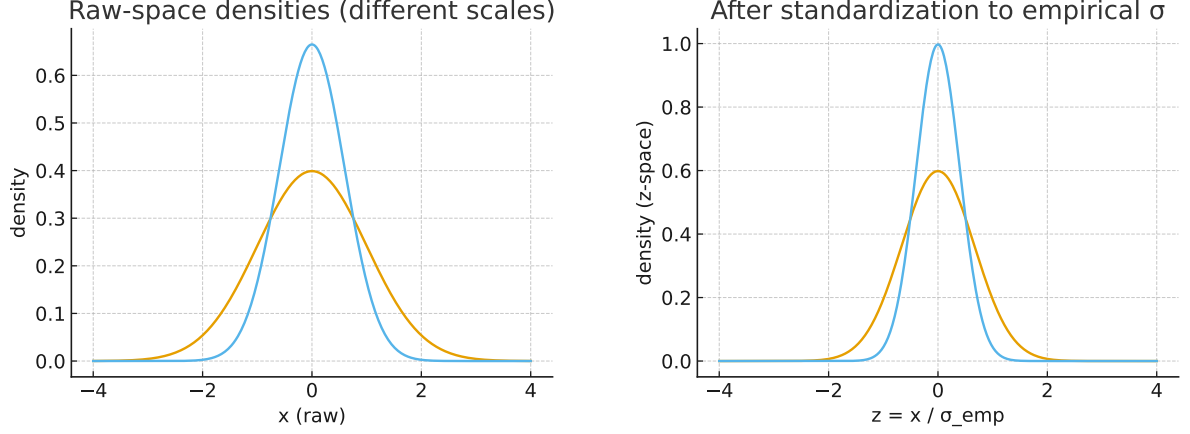


Figure 3: Demonstration of the standardization procedure. **Left:** Raw-space probability density functions for empirical market data (blue) and a theoretical Generalized Hyperbolic distribution (orange) showing significant scale mismatch that complicates direct comparison. **Right:** The same distributions after standardization to empirical- σ units, where both curves operate on the same scale enabling meaningful CDF comparisons and direction statistics. The standardization preserves distributional shapes while eliminating scale-dependent artifacts.

Values $p_{\text{dir}} > 0.5$ indicate rightward mass shift in the Hamiltonian prediction relative to current empirical distribution, suggesting potential upward price movement.

5.3 Cost-Aware Magnitude Gating

Trading costs (spreads, fees, slippage) are explicitly incorporated through exceedance probability calculations. Let c_{raw} denote total trading costs in return units. The standardized cost threshold becomes:

$$c_z = \frac{c_{\text{raw}}}{\sigma_{\text{emp}}} \quad (21)$$

We compute the exceedance probability:

$$\pi(c_z) = \mathbb{P}(|Z_{t+1}| > c_z) \quad (22)$$

Trading occurs only when $\pi(c_z) \geq \theta$ for some threshold $\theta \in [0.6, 0.7]$, ensuring sufficient expected movement to overcome trading costs.

6 Theoretical Properties

6.1 Energy Conservation and Stability

Theorem 1 (Energy Conservation). *In the absence of dissipation, the Hamiltonian $H(q, p)$ is conserved along solution trajectories of equations (11)-(12).*

Proof. Direct calculation shows $\frac{dH}{dt} = \frac{\partial H}{\partial q} \cdot \frac{dq}{dt} + \frac{\partial H}{\partial p} \cdot \frac{dp}{dt} = 0$ using the canonical equations. $\square$

Theorem 2 (Domain Invariance). *The barrier potential $V_{\text{bar}}(q)$ ensures that finite-energy trajectories remain within the physically meaningful domain $\{\sigma > \sigma_{\text{min}}, \text{kurt} > 1\}$.*

Proof. As $\sigma \rightarrow \sigma_{\text{min}}^+$ or $\text{kurt} \rightarrow 1^+$, we have $V_{\text{bar}}(q) \rightarrow +\infty$, making such configurations energetically inaccessible for finite total energy. $\square$

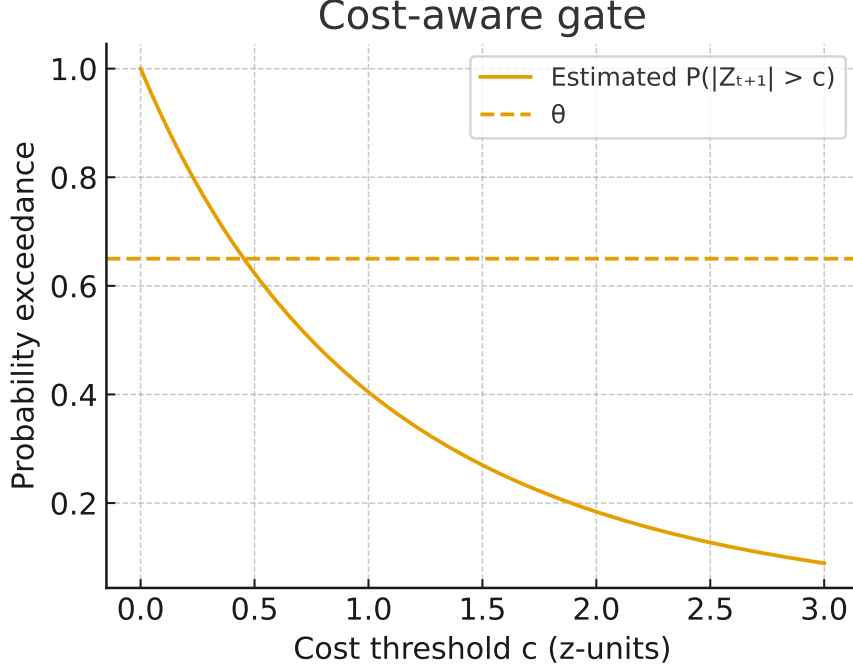


Figure 4: Cost-aware magnitude gating mechanism. The plot shows the exceedance probability $\pi(c) = \mathbb{P}(|Z_{t+1}| > c)$ as a function of the standardized cost threshold c . The horizontal dashed line represents the fixed gating threshold θ . Trading signals are only acted upon when the predicted probability of exceeding trading costs surpasses this threshold, ensuring that expected price movements are sufficient to overcome transaction frictions. This approach directly incorporates trading realities into the decision-making process without requiring separate risk management overlays.

6.2 Scale Invariance of Direction Statistic

Lemma 1 (Scale Invariance). *The direction statistic p_{dir} defined in equation (20) is invariant under positive rescaling of the underlying returns, provided both empirical and theoretical distributions are standardized consistently.*

Proof. Under the transformation $X \mapsto aX$ for $a > 0$, both CDFs F_e and F_h undergo the same monotonic transformation when evaluated at the correspondingly scaled pivot point, preserving their difference. $\square$

Lemma 2 (Monotonicity of Exceedance). *The exceedance probability $\pi(c) = \mathbb{P}(|Z| > c)$ is non-increasing in c , with $\pi(0) = 1$ and $\lim_{c \rightarrow \infty} \pi(c) = 0$.*

Proof. For fixed distribution of Z , the sets $\{|Z| > c\}$ are nested decreasing in c . $\square$

7 Implementation Considerations

7.1 Temporal Adaptation

The system incorporates adaptive time-stepping based on characteristic timescales and energy levels. The integration step size is adjusted according to:

$$\Delta t = \min \left(\Delta t_{\max}, \frac{\tau_{\text{char}}}{\sqrt{1 + H/H_{\text{ref}}}} \right) \quad (23)$$

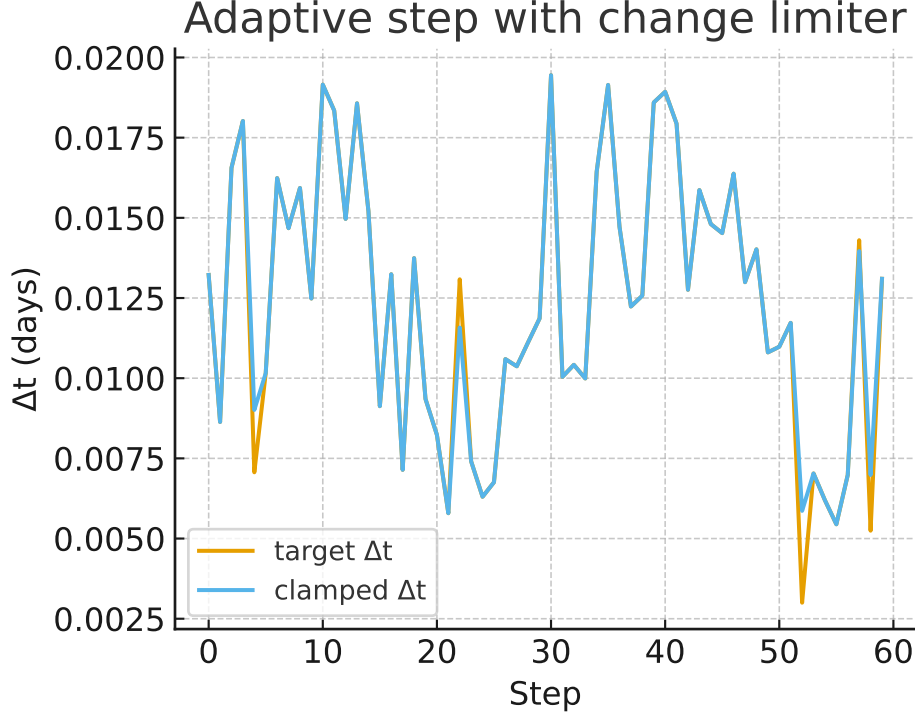


Figure 5: Adaptive time-stepping with change limiting. The plot shows the target integration step size (thin line) and the actual clamped step size (thick line) over a sequence of integration steps. The change limiter prevents excessive step size variations that could destabilize the numerical integration, while still allowing adaptation to changing system dynamics. This mechanism ensures numerical stability while maintaining computational efficiency by taking larger steps during quiescent periods and smaller steps during rapid evolution phases.

where τ_{char} represents the fastest characteristic timescale and H_{ref} is a reference energy scale.

7.2 Momentum Freshness

To account for the decreasing relevance of momentum over time, we introduce a freshness factor:

$$\text{freshness} = \exp(-\text{age}/\tau_{\text{char}}) \cdot \sigma(\text{kinetic fraction}) \cdot \sigma(\text{SNR}) \quad (24)$$

where "age" tracks time since momentum computation, and the additional factors account for energy distribution and signal quality.

7.3 Convergence Diagnostics

RMHMC chain convergence is monitored through standard diagnostics including the Gelman-Rubin $\hat{R}$ statistic and effective sample size calculations. Multiple independent chains with different initializations ensure robust sampling.

8 Model Validation and Sanity Checks

The framework includes several internal consistency checks:

1. **Normalization:** Predictive and empirical densities integrate to 1 ± 10^{-3} on the shared grid

2. **Domain constraints:** Statistical validity requirements are maintained throughout evolution
3. **No-arbitrage monitoring:** Triangle penalty magnitude flags potential market dislocations
4. **Sampler diagnostics:** Chain agreement and absence of divergences in RMHMC

9 Discussion and Future Directions

9.1 Advantages of the Hamiltonian Approach

The physics-based framework offers several conceptual and practical advantages:

Theoretical Grounding: Unlike purely statistical models, the Hamiltonian approach provides a principled framework for understanding parameter evolution based on established physical laws.

Robustness: By avoiding direct curve fitting and focusing on distributional properties, the method may be less susceptible to overfitting and regime changes.

Interpretability: The energy formulation provides intuitive understanding of market dynamics: high kinetic energy indicates rapid statistical change, while potential wells represent stable regimes.

Extensibility: The framework naturally accommodates additional physics-inspired features such as cross-asset energy transfer or ensemble forecasting.

9.2 Limitations and Challenges

Dimensionality: The approach trades granularity for robustness by operating on low-dimensional moment statistics rather than full price trajectories.

Standardization Assumptions: The method assumes local stability of empirical σ for standardization purposes, which may break down during extreme market events.

Computational Complexity: RMHMC sampling requires careful tuning and may be computationally intensive for real-time applications.

Parameter Calibration: While avoiding traditional curve fitting, the framework still requires specification of energy parameters and thresholds.

9.3 Future Research Directions

Several extensions merit investigation:

Multi-Asset Energy Systems: Modeling currency pairs as components of a unified energy system with cross-pair energy transfer mechanisms.

Non-Equilibrium Thermodynamics: Incorporating dissipation and external driving forces to model trending markets.

Quantum Extensions: Exploring quantum mechanical analogies for incorporating uncertainty and measurement effects.

Ensemble Methods: Generating multiple trajectories to quantify forecast uncertainty and improve decision robustness.

10 Conclusion

We have presented a novel Hamiltonian framework for foreign exchange modeling that treats statistical moments as dynamical variables evolving according to physically motivated laws. By standardizing to empirical- σ units and employing distribution-level decision criteria, the

approach avoids traditional curve fitting while explicitly accounting for trading costs and no-arbitrage constraints.

The framework represents a paradigmatic shift from prediction-based to mechanism-based market modeling. Rather than attempting to forecast specific price movements, it focuses on understanding how distributional properties evolve over time according to conservation laws and energy principles.

While preliminary, this work establishes theoretical foundations for a new class of physics-inspired trading systems. The combination of rigorous mathematical structure, practical implementation considerations, and explicit cost accounting suggests potential for robust performance across varying market conditions.

Future empirical validation will determine whether this theoretical promise translates to practical trading advantages. However, the framework’s emphasis on physical principles over fitted parameters provides reason for optimism regarding its robustness and generalizability.

A Notation Summary

Symbol	Meaning
q, p	Moment state and conjugate momentum vectors
q^{eq}	Equilibrium moment state
K	Stiffness matrix for mean reversion
T, V, H	Kinetic, potential, and total energy
Z	Standardized return coordinate X/σ_{emp}
F_e, F_h	Empirical and Hamiltonian CDFs
p_{dir}	Direction statistic for trading decisions
c_z	Standardized cost threshold
$\pi(c_z)$	Exceedance probability
θ	Cost gating threshold
τ_{char}	Characteristic timescale
Δ	Cross-coupling matrix
η	No-arbitrage penalty strength

Table 1: Key notation used throughout the paper.

B Key Theoretical Results

Lemma 1 (Energy Conservation). *In the absence of dissipation, the Hamiltonian $H(q, p)$ is conserved along solution trajectories of equations (11)-(12).*

Lemma 2 (Domain Invariance). *The barrier potential $V_{\text{bar}}(q)$ ensures that finite-energy trajectories remain within the physically meaningful domain $\{\sigma > \sigma_{\text{min}}, \text{kurt} > 1\}$.*

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